BOUNDED RATIONALITY: MODELS FOR SOME FAST AND FRUGAL HEURISTICS

HORACIO ARLÓ COSTA AND ARTHUR PAUL PEDERSEN

ABSTRACT. We generalize two versions of Gerd Gigerenzer's Take The Best algorithm to the case of non-binary choice and we study choice functions that completely characterize the algorithm. One of the versions of the algorithm allows for failures of transitivity and acyclicity of the preference relation induced by the algorithm. So, the corresponding choice function has a descriptive interpretation (it violates the usual constraint that choices over non-empty sets yield non-empty outputs).

1. INTRODUCTION

Herb Simon pioneered the study of *bounded* models of rationality. Simon famously argued that decision makers typically *satisfice* rather than *optimize*. According to Simon, a decision maker normally chooses an alternative that meets or exceeds specified criteria, even when this alternative is not guaranteed to be unique or in any sense optimal. For example, Simon argued that an organism—instead of scanning all the possible alternatives, computing each probability of every outcome of each alternative, calculating the utility of each alternative, and thereupon selecting the optimal option with respect to expected utility—typically chooses the first option that satisfies its 'aspiration level.'

Simon insisted that real minds are *adapted* to real-world environments. Simon writes, "Human rational behavior is shaped by a scissors whose two blades are the structure of task environments and the computational capabilities of the actor" [18, p. 7]. Yet by paying undivided attention to the heuristics used by real agents, a very influential line of work on models of bounded rationality privileges one of these two blades over the other. This is the tradition initiated by the heuristics-and-biases program championed by Kahneman and Tversky (see the classic [11] and the more recent [10]). According to followers of this tradition, laws of probability, statistics, and logic constitute normative laws of human reasoning; descriptively, nevertheless, human reasoners follow heuristics that are systematically biased and error-prone. But this is hardly Simon's idea, who insisted on the accuracy of adapted and bounded methods.

Gerd Gigerenzer and his research group in Germany have followed Simon's view about bounded rationality more closely than many other researchers also deriving inspiration from Simon. The following gives an idea of Gigerenzer's main interests and goals:

> We wish to do more than just oppose the Laplacian demon view. We strive to come up with something positive that could replace this unrealistic view of the mind. What are these simple, intelligent heuristics capable of making near-optimal inferences? How fast and how accurate are they?...[W]e propose a class of heuristics that exhibit bounded rationality in both of Simon's senses [this is a reference to Simon's scissors metaphor]. These 'fast and frugal' heuristics operate with simple psychological principles that satisfy the constraints of limited time, knowledge and computational might, rather than those of classical rationality. At the same

> time they are designed to be fast and frugal without a significant loss of inferential accuracy because they can exploit the structure of the environments.

A way to reconcile this view with the normative status of laws of logic and statistics is to argue that the cognitive role of the fast and frugal heuristics is to approximate the standards of these laws: When the methods in question are sufficiently well adapted to the environment, we will have a happy convergence on recommendations of these normative laws. This is, for example, Dawkins' view of the 'rules of thumb' converging on optimality provided by natural selection [5]. Indeed, this is a nice way of interpreting the role of the fast and frugal heuristics that Gigerenzer discusses. But Gigerenzer has quite emphatically denied that this is his view. In fact, Gigerenzer has argued that it is rational to use his fast and frugal heuristics even when they conflict with the dictates of the traditional conception of rationality.

In this article we focus on one fast and frugal heuristic Gigerenzer advertises and that he calls Take The Best. There are at least two varieties of the heuristic, defined in [7] and [8]. So more precisely, there are two heuristics, and we will investigate these heuristics in this article. One goal of this article is a study of the choice-theoretical properties of these heuristics. In order to do so, both varieties of Take The Best have to be extended beyond two-alternative-choice tasks, presently the primary application of Gigerenzer's fast and frugal heuristics. We provide here a first approximation to this problem. Our strategy is to extend choice beyond two-alternative-choice tasks by maximizing the binary preference relations induced by both varieties of Take The Best. This permits the study of abstract conditions of choice that completely characterize both varieties. The methodology employed extends techniques that are commonly used in the theory of choice functions.

The characterization results presented in this article are complemented by a different model offered in a companion article [3]. In this article we consider an extension that is psychologically more plausible (in the sense that it only appeals to satisficing methods).

We will see that one version of Take The Best violates some basic rationality constraints like transitivity and that important axiomatic properties of choice functions are violated as well. We will also see that the other variant of the heuristic is fairly well behaved and compatible with some salient norms of rationality and axiomatic properties of choice. Accordingly, we claim that speed and frugality can be obtained without renouncing basic norms of rationality. Gigerenzer seems to argue in various places against this view, suggesting that speed and frugality cannot be reconciled with the traditional norms of rationality. Perhaps his claim can be reconstructed as asserting that *psychologically plausible* fast and frugal heuristics cannot be reconciled with norms of rationality. But even in this case there is a general theory of choice functions that can be developed for these fast and frugal heuristics. Of course, the interpretation of the choice functions used in the characterization of (both varieties of) Take The Best is not the usual one. The idea is to develop an empirical (descriptive) theory of choice functions. The main interpretation of axiomatic properties of choice is not that of imposing normative constraints on choice, but as describing patterns of inference permitted by the underlying heuristic, which is taken as an epistemological primitive.

2. TAKE THE BEST

The best manner of making clear Gigerenzer's position is by means of an example. Consider the heuristic Gigerenzer and his collaborators call Take The Best. The main idea behind this heuristic is that for a given domain of objects X, an agent decides between two objects with respect to a set of *cues* ordered by *cue validity*. We can make this idea more precise as follows.

Definition 2.1. Let X be a collection of objects.

(i) A *cue* for X is a function $Cue : X \to \{-, ?, +\}$,
(ii) A *recognition heuristic* for X is a function $Rec : X \to \{-, +\}$. If $Rec(a) = +$, we say that a is *recognized*.
(iii) Let Y be the collection of all subsets of X of cardinality 2. A *randomizer* for X is a function $Ran : Y \to X$ such that $Ran(A) \in A$ for each $A \in Y$. If $Ran(\{a, b\}) = a$, we say that a is *chosen at random*.

Definition 2.2. Let $(Cue_i)_{i<n}$ be a finite collection of cues for X, where Cue_0 is a recognition heuristic. We call the pair $(X, (Cue_i)_{i<n})$ a *cue validity ordering* over X (and say that $(Cue_i)_{i<n}$ is *ordered by cue validity*) if for every $a \in X$, if $Cue_0(a) = -$, then for each i with $0 < i < n$, $Cue_i(a) =?$.

Definition 2.3. If $\mathfrak{C}$ is a cue validity ordering over a set X and Ran is a randomizer for X, we call the pair $(\mathfrak{C}, Ran)$ a *Take The Best frame* over X.

We denote Cue_0 by Rec. Intuitively, $(Cue_i)_{i<n}$ is ordered according to the reliability of the cues, a recognizer is nothing more than a cue with the demand that its range is $\{-, +\}$, and a randomizer is employed to make a guess when no cue provides sufficient information to make a choice between a pair of objects, i.e., when no cue discriminates or is decisive between a pair of objects.

The following algorithm for binary choices with respect to a Take The Best frame over X comprises the Take The Best heuristic.

> **Step 0**: *Recognition Heuristic*. If only one of two possible objects is recognized, then choose the recognized object. If neither of the two objects is recognized, then choose randomly between them. If both of the objects are recognized, then proceed to Step 1.
> **Step 1**: *Search Rule*. Choose the cue with the highest validity that has not yet been tried for this choice task. Look up the cue values of the two objects.
> **Step 2**: *Stopping Rule*. If one object has a positive cue value and the other does not, then stop the search and go to Step 3. Otherwise go back to Step 1 and search for another one. If no further cue is found then guess.
> **Step 3**: *Decision Rule* (*one-reason decision making*). Predict that the object with the positive cue value has the higher value on the criterion.

Step 0 demands that if $Rec(a) \neq Rec(b)$, then one should choose the object from $\{a, b\}$ to which Rec assigns a positive value. But if $Rec(a) = Rec(b) = -$, then one should choose the object from $Ran(\{a, b\})$. In the remaining case when $Rec(a) = Rec(b) = +$, one should proceed to Step 1. Observe that in the case that $Rec(a) = Rec(b) = -$, it follows that for each i with $0 < i < n$, $Cue_i(a) = Cue_i(b) =?$.

Hence, Step 0, Step 1, Step 2, and Step 3 together require one to search for the least $i < n$ for which $Cue_i(a) = +$ and $Cue_i(b) \in \{-, ?\}$ or $Cue_i(b) = +$ and $Cue_i(a) \in \{-, ?\}$, whereby the object from $\{a, b\}$ to which Cue_i assigns a positive value is chosen. However, if there is no such i, then one should choose the object from $Ran(\{a, b\})$.

Observe that search operates only on a fraction of the total knowledge in memory and is stopped when a cue discriminates between a pair of objects. Thus, Take The Best is similar to Simon's satisficing algorithm insofar as it terminates search when it finds the first cue that discriminates between a pair of objects and thereupon makes a choice. Also observe that the algorithm does not integrate information, reducing all aspects of information to a single dimension; rather, it employs one-reason decision making, thereby conflicting with the classical economic view of human behavior.

Gigerenzer considers two-alternative-choice tasks in various contexts where inferences on a quantitative dimension must be made from memory under the constraints of limited time and knowledge. Gigerenzer offers various examples of two-alternative-choice tasks in [7]. As an illustration, consider the following prompt:

> Which city has a larger population?
> (a) Hamburg
> (b) Cologne

Now suppose that an agent neither knows nor can deduce the answer to this question and so must make an inductive inference based on related real-world knowledge. Thus, the problem is how such an inference can be made. According to the theory of *probabilistic mental models* that Gigerenzer adopts, the first step is to search knowledge about a reference class, e.g., 'Cities in Germany.' The knowledge itself consists of probability cues and their corresponding values for the objects in the reference class. For example, a candidate for a cue may be a function of whether a city has a professional soccer team in the major league. According to this cue, if one city has a team in the major league and the other does not, the city with the professional soccer team in the major league is likely—but perhaps not certain—to have a larger population than the other city. Furthermore, according to Gigerenzer's Take The Best heuristic, if this cue is the first cue to discriminate between the two cities, the city with the professional soccer team is inferred to have a larger population than the other city. Other examples of cues for the population demographics task are functions of answers to the following questions: (*Intercity Train*) 'Is the city on the Intercity line?'; (*University*) 'Is the city home to a University?'; (*Industrial Belt*) 'Is the city in the Industrial Belt?'

Gigerenzer utilizes probabilistic mental models that are sensible to the fact that an agent typically does not know all the information upon which it could base an inference. These probabilistic mental models reflect at least two aspects of such limited knowledge: On the one hand, an agent can have incomplete knowledge of the objects in the relevant reference class; on the other hand, the person might have limited knowledge of values of cues.

We have seen that each cue has an associated cue validity. The validity of a cue specifies the predictive reliability or predictive power of the cue. More precisely, the *ecological validity* of a cue is the relative frequency (with respect to a given reference class) that the cue correctly predicts the target variable (e.g., population). Returning to the aforementioned example, in 87 percent of the pairs in which one city has a professional soccer team and the other does not, the city with the professional soccer team has a larger population. Thus, .87 specifies the ecological validity of the soccer team cue (see [7, p. 176]). In Gigerenzer's model, it is required that there are only finitely many cues and that no two cues have the same ecological validity.

In Gigerenzer's original article [7, pp. 125-163], Take The Best has a different stopping rule. In this earlier version, search terminates only when one object has a positive value and the other has a negative value:

> **Step 2′**: *Stopping Rule*. If one object has a positive cue value and the other has a negative value, then stop the search and go to Step 3. Otherwise go back to Step 1 and search for another one. If no further cue is found then guess.

According to Gigerenzer, the Take The Best heuristic presented above follows empirical evidence that agents tend to use the fast, simpler stopping rule expressed in Step 2 [7, p.

175]. Nonetheless, we will consider both varieties of the heuristic in this article. We will thereby call Gigerenzer's original heuristic (with Step 2′ instead of Step 2) the Original Take The Best heuristic.

Definition 2.4. Let $(X, (Cue_i)_{i<n}, Ran)$ be a Take The Best frame over X. For each $i < n$, define a binary relation $\succ$ over X by setting for all $a, b \in X$,

$$a \succ_i b \text{ if and only if } Cue_i(a) = + \text{ and } Cue_i(b) = -.$$

Similarly, for each $i < n$, define a binary relation $>$ over X by setting for all $a, b \in X$,

$$a >_i b \text{ if and only if } Cue_i(a) = + \text{ and } Cue_i(b) \in \{?, -\}.$$

For each $i < n$, we say that Cue_i is *decisive* for $a, b \in X$ if $a \succ_i b$ or $b \succ_i a$. We also call Cue_i *discriminating* for $a, b \in X$ if $a >_i b$ or $b >_i a$.

We are now in a position to define the notion of a Take The Best model with respect to a Take The Best frame.

Definition 2.5. Let $(X, (Cue_i)_{i<n}, Ran)$ be a Take The Best frame over X. Define a binary relation $\succ$ over X by setting for every $a, b \in X$,

$$a \succ b \quad \text{if and only if} \quad \begin{array}{l} a \succ_i b \text{ for the least } i < n \text{ for which } Cue_i \text{ is decisive for } a, b, \\ \text{or there is no decisive cue for } a, b \text{ and } Ran(\{a, b\}) = a. \end{array}$$

Similarly, define a binary relation $>$ over X by setting for $a, b \in X$,

$$a > b \quad \text{if and only if} \quad \begin{array}{l} a >_i b \text{ for the least } i < n \text{ for which } Cue_i \text{ is discriminating for } a, b, \\ \text{or there is no discriminating cue for } a, b \text{ and } Ran(\{a, b\}) = a. \end{array}$$

We thereby call the quadruple $(X, (Cue_i)_{i<n}, Ran, \succ)$ an *Original Take The Best model*. Similarly, we call the quadruple $(X, (Cue_i)_{i<n}, Ran, >)$ a *Take The Best model*.

Definition 2.6. Let $(X, (Cue_i)_{i<n}, Ran, \succ)$ be an Original Take The Best model, and let $(X, (Cue_i)_{i<n}, Ran, >)$ be a Take The Best model.

(i) We say that $\succ$ is *decisive* if there is a decisive cue for every distinct $a, b \in X$.
(ii) We say that $>$ is *discriminating* if there is a discriminating cue for every distinct $a, b \in X$.
(iii) We call $(X, (Cue_i)_{i<n}, Ran, \succ)$ *decisive* (asymmetric, connected, transitive, modular, etc.) if $\succ$ is decisive (asymmetric, connected, transitive, modular, etc.).
(iv) We call $(X, (Cue_i)_{i<n}, Ran, >)$ *discriminating* (asymmetric, connected, transitive, modular, etc.) if $>$ is discriminating (asymmetric, connected, transitive, modular, etc.).

Observe that by definition both $\succ$ and $>$ are asymmetric (and so irreflexive) and connected.

As Gigerenzer points out, both the Original Take The Best heuristic and the Take The Best heuristic are hardly standard statistical tools for inductive inference: they do not use all available information and are non-linear. Furthermore, the binary relations for models of both heuristics may violate transitivity if they fail to be discriminating or decisive. But an Original Take The Best model may violate transitivity even if it is decisive. Let's see an example of the latter feature.

Example 2.7. Consider a collection X of three objects, *a*, *b* and *c*, a recognition heuristic Rec, and three cues, Cue_1, Cue_2, and Cue_3 (any randomizer will do), as presented in the table below:

	a	b	c
Rec	$+$	$+$	$+$
Cue_1	$+$	$-$	$?$
Cue_2	$-$	$+$	$-$
Cue_3	$-$	$-$	$+$

Clearly, $a \succ b$ in view of the decisive cue Cue_1 for a and b. Also, $b \succ c$ in light of the decisive cue Cue_2 for a and b. But according to the third cue, $c \succ a$.

□

Not only does this constitute a violation of transitivity, but it also induces a cycle of preference. So an Original Take The Best model may violate various salient norms of rationality, even if there is no guessing involved, i.e., even if it is decisive.

The first thing that one can say in defense of Take The Best is that it seems descriptively adequate.[1] This is interesting *per se*, but this does not indicate anything regarding other potential cognitive virtues of the heuristic. It is at this point that Gigerenzer turns to the ecological aspect of the argument for bounded rationality. Gigerenzer contends that if fast and frugal heuristics are well-tuned ecologically, then they should not fail outright. So in [7], Gigerenzer proposes a competition among Take The Best and other inferential methods which are more costly but which integrate information. These rival methods (like linear regression) obey norms of rationality that Take The Best could violate. The idea of the competition is that the contest will go to the strategy that scores the highest proportion of correct inferences (accuracy) using the smallest number of cues (frugality).

And in [7] Gigerenzer presents surprising results showing that Take The Best wins the contest. All results are computed for simulations taking into account empirical constraints that are determined by the results of behavioral experiments.

As Bermúdez points out in [4], Gigerenzer claims to have replaced criteria of rationality based upon logic and probability theory with a heuristic-based criteria of real-world performance. There is an innocuous way of interpreting such a claim via a 'long run' argument. One might claim that it is beneficial to stick to certain heuristics given the long run benefits it provides, even if in the short run it might lead you astray. But this does not seem to be the argument that Gigerenzer sets forth for his heuristics. Apparently he wants to argue that something more controversial than this flows from the result of the competition between different inferential methods (after all, the long run argument still presupposes the normativity of laws of logic and statistics). The fast and frugal heuristics can trump normative theories when adapted to the ecological environment.

Simon was centrally interested in discovering the inferential strategies that humans use in the real world. His interest was not to provide a new theory of rationality but to engage in psychology. It is unclear the extent to which Gigerenzer might be interested in replacing standard norms of rationality with fast and frugal strategies of inference that, when adapted, are also accurate and successful. Undoubtedly, the strategies in question remain interesting under a purely descriptive point of view. In this article we propose to begin a study of the theory of choice that applies to these methods. In a companion article we intend to apply this theory of choice to study the structure of bounded methods of belief change.

[1]See [7, pp. 174-175] for an account of the empirical evidence supporting Take The Best.

3. Rational Choice: the received view

One of the central tools in the theory of rational choice is the notion of *choice function*. Here we present this notion, largely following the classical presentations of Amartya Sen [16] and Kotaro Suzumura [19].

Definition 3.1. Let X be set, and let $\mathscr{S}$ be a collection of non-empty subsets of X. We call the pair $(X, \mathscr{S})$ a *choice space*. A *choice function* on $(X, \mathscr{S})$ is a function $C : \mathscr{S} \to \mathscr{P}(X)$ such that $C(S) \subseteq S$ for every $S \in \mathscr{S}$.

For a choice function C on a choice space $(X, \mathscr{S})$, we call $C(S)$ the *choice set* for S. Intuitively, a choice function C selects the 'best' elements of each S, and $C(S)$ represents the 'choosable' elements of S.

When the context is clear, we will often talk of a choice function C on a choice space $(X, \mathscr{S})$ without reference to the choice space. Thus, we may simply speak of a choice function C. For our purposes it is convenient to restrict our attention to choice functions each of which has a domain that is the family of all nonempty finite subsets of a collection of objects. We wish to emphasize, however, that many of the results presented in the remainder of this article are preserved when less restrictive conditions are imposed on the domains of choice functions.

The following condition demarcates a special class of choice functions. Often in the literature it is assumed that choice functions satisfy this condition.

Regularity. For each $S \in \mathscr{S}$, $C(S) \neq \emptyset$.

We call a choice function satisfying Regularity *regular*.

The idea that a choice function C picks the 'best' elements of each $S \in \mathscr{S}$ has been made more precise by assuming that there is some binary relation over the elements of X according to which $C(S)$ distinguishes the best elements of each $S \in \mathscr{S}$. A choice function with this property is called a *rational* choice function. Two formalizations of the idea of a rational choice function have been widely utilized in the literature. We discuss these formalizations here.

The first formalization of a rational choice function demands that a choice function C selects the optimal (greatest) elements from each set $S \in \mathscr{S}$. Here we need some notation. The set of greatest elements of a set S with respect to a binary relation $\geq$ is defined as follows:

$$G(S, \geq) := \{x \in S : x \geq y \text{ for all } y \in S\}$$

Using this notation, we now offer a definition.

Definition 3.2. A binary relation $\geq$ on a universal set X *G-rationalizes* (or is a *G- rationalization* of) a choice function C on a choice space $(X, \mathscr{S})$ if for every $S \in \mathscr{S}$, $C(S) = G(S, \geq)$.

We also say that a choice function C is *G-rational* if there is a binary relation $\geq$ that G-rationalizes C. This formalization of a rational choice function is what Sen and others often call an *optimizing* notion of rationality.

The second formalization of a rational choice function captures a less stringent notion of rationality, demanding only that a choice function selects the *maximal* elements from each set $S \in \mathscr{S}$. Again, we require some notation. The set of maximal elements of a set S with respect to a binary relation $>$ is defined as follows:

$$M(S, >) := \{x \in S : y > x \text{ for no } y \in S\}$$

With this notation at hand, we again offer a definition.

Definition 3.3. A binary relation $>$ on a universal set X *M-rationalizes* (or is a *M-rationalization* of) a choice function C on a choice space $(X, \mathscr{S})$ if for every $S \in \mathscr{S}$, $C(S) = M(S, >)$.

As with G-rationality, we say that a choice function C is *M-rational* if there is a binary relation $>$ that M-rationalizes C. Sen and others call this a *maximizing* notion of rationality. We will assume this notion of rationality in this article.[2]

In the following, let $\mathscr{P}_{fin}(X)$ denote the family of all finite subsets of a collection of objects X. The next proposition establishes that any reasonable binary relation that M-rationalizes a choice function on a choice space $(X, \mathscr{P}_{fin}(X))$ is unique.

Proposition 3.4. *Let C be a choice function on a choice space $(X, \mathscr{P}_{fin}(X))$. Then if $>$ is a irreflexive binary relation on X that M-rationalizes C, $>$ uniquely M-rationalizes C.*

Proof. Let $>$ be a irreflexive binary relation on X that M-rationalizes C, and let $>'$ be a binary relation on X that also M-rationalizes C. Now since $>$ is irreflexive, $C(\{x\}) = \{x\}$ for every $x \in X$, whereby it follows that $>'$ is irreflexive. Now for *reductio ad absurdum*, suppose $> \neq >'$. Without loss of generality, we may assume that for some $x, y \in X$, $x > y$ but $x \not>' y$. Then $y \notin M(\{x, y\}, >)$, but $y \in M(\{x, y\}, >')$, yielding a contradiction. □

Remark 3.5. Observe that the irreflexivity of $>$ is a necessary condition for Proposition 3.4. To illustrate, consider a choice space $(X, \mathscr{P}_{fin}(X))$, where $X := \{x, y\}$, and choice function C on $(X, \mathscr{P}_{fin}(X))$ defined by setting $C(\{x\}) := \emptyset$, $C(\{y\}) = \{y\}$, and $C(X) := \{y\}$. We can define two binary relations $>$ and $>'$ by setting $> := \{(x, x)\}$ and $>' := \{(x, x), (y, x)\}$. Clearly neither $>$ nor $>'$ is irreflexive, both $>$ and $>'$ M-rationalize C, yet $> \neq >'$.

We call a choice function C *acyclic* (asymmetric, connected, transitive, modular, etc.) M*-rational* if C is M-rationalized by an acyclic (asymmetric, connected, transitive, modular, etc.) binary relation. Using this terminology we can state the following lemma with ease.

Lemma 3.6 ([19], p. 35)**.** *A choice function C on a choice space $(X, \mathscr{P}_{fin}(X))$ is acyclic M-rational if and only if it is regular M-rational.*

Proof. For the direction from left to right, suppose C is acyclic M-rational. We must show that C is regular. For *reductio ad absurdum*, assume that C is not regular, and let $S \in \mathscr{P}_{fin}(X)$ be such that $C(S) = \emptyset$. Choose $x_0 \in S$. Since S is finite, there are $x_1, \ldots, x_{n-1} \in S$ such that $x_0 > x_{n-1} > \cdots > x_1 > x_0$, contradicting that C is acyclic M-rational.

For the direction from right to left, suppose C is regular M-rational, and let $>$ be an M-rationalization of C. For *reductio ad absurdum*, assume that $>$ is not acyclic. Then there are $x_0, \ldots, x_{n-1} \in X$ such that $x_0 > x_{n-1} > \cdots > x_1 > x_0$. But then $C(\{x_0, \ldots, x_{n-1}\}) = M(\{x_0, \ldots, x_{n-1}\}, >) = \emptyset$, contradicting that C is regular. □

There has been some work on the project of characterizing the notion of rationality axiomatically using so-called *coherence constraints*. One salient coherence constraint is the condition Sen calls *Property* α [16, p. 313], also known as *Chernoff's axiom* [19, p. 31].

Property α. For all $S, T \in \mathscr{S}$, if $S \subseteq T$, then $S \cap C(T) \subseteq C(S)$.

---

[2]It is well known that if we require that $\geq$ be understood as the asymmetric part of $>$, then *an M-rational choice function is G-rational, but not vice-versa in general*.

There are two lines of argument for the characterization of rationality, one proposed by Sen in [16] (and reconsidered in [17]) and another proposed by Kotaro Suzumura in [19]. Both use the notion of *base preference* ([16, p. 308], [17, p. 64], [19, p. 28]). We modify their respective arguments in terms of maximization.

Definition 3.7 (Base Preference). Let C be a choice function on a choice space $(X, \mathscr{P}_{fin}(X))$. We define (*strict*) *base preference* by setting

$$>^C := \{(x, y) \in X \times X : x \in C(\{x, y\}) \text{ and } y \notin C(\{x, y\})\}$$

Observe that $>^C$ must be asymmetric and so irreflexive.

We now present Suzumura's line of argument. As a first step we can add a new coherence constraint [19, p. 32] which we have reformulated in terms of maximization:

Generalized Condorcet Property. For all $S \in \mathscr{S}$, $M(S, >^C) \subseteq C(S)$

The first general result that Suzumura presents is the following (modified for our purposes):

Theorem 3.8 ([19], p. 35)**.** *A choice function C on a choice space $(X, \mathscr{P}_{fin}(X))$ is acyclic M-rational if and only if it is regular and satisfies Property α and the Generalized Condorcet Property.*

Proof. By Lemma 3.6, it suffices to show that a regular choice function C on a space $(X, \mathscr{P}_{fin}(X))$ is M-rational if and only if it satisfies Property α and the Generalized Condorcet Property.

($\Rightarrow$) Suppose C is M-rational, and let $>$ be a M-rationalization of C.
Property α. Let $S, T \in \mathscr{P}_{fin}(X)$. If $S \subseteq T$ and $x \in S \cap C(T)$, then $y > x$ for no $y \in T$, whence $y > x$ for no $y \in S$, so $y \in C(S)$.
Generalized Condorcet Property. Let $S \in \mathscr{P}_{fin}(X)$, and suppose $x \in M(S, >^C)$. Then because C is regular, $x \in C(\{x, y\})$ for all $y \in S$, whereby $y > x$ for no $y \in S$ and so $x \in C(S)$.

($\Leftarrow$) Suppose C satisfies Property α and the Generalized Condorcet Property. We must show that $>^C$ M-rationalizes C. By the Generalized Condorcet Property, we must only verify that $C(S) \subseteq M(S, >^C)$ for all $S \in \mathscr{P}_{fin}(X)$. So let $S \in \mathscr{P}_{fin}(X)$, and suppose $x \in C(S)$. Then by Property α, for every $y \in S$, $x \in C(\{x, y\})$ and so $y \not>^C x$, whereby $x \in M(S, >^C)$, as desired.

□

The proof relies heavily on the assumption that choice functions are to pick *non-empty* subsets of the sets from which they make selections. So Suzumura's axiomatic characterization of rationality depends in both cases on the use of *regular* choice functions.

We now present the second line of argument for the characterization of rationality due to Sen [16]. The argument also proceeds in terms of regular choice functions and (as indicated above) uses the notion of base preference. We need an additional axiom in order to present the argument:

Property γ. For every nonempty $I \subseteq \mathscr{S}$ such that $\bigcup_{S \in I} S \in \mathscr{S}$,

$$\bigcap_{S \in I} C(S) \subseteq C(\bigcup_{S \in I} S)$$

With this we can state the following theorem:

Theorem 3.9. *A choice function C on a choice space $(X, \mathscr{P}_{fin}(X))$ is acyclic M-rational if and only if it is regular and satisfies Property α and Property γ.*

Proof. By Lemma 3.6, it suffices to show that a regular choice function C on a space $(X, \mathscr{P}_{fin}(X))$ is M-rational if and only if it satisfies Property α and Property γ.

($\Rightarrow$) Suppose C is M-rational, and let $>$ be a M-rationalization of C. In light of the proof of Theorem 3.8, we only show that C satisfies Property γ. Let $I \subseteq \mathscr{P}_{fin}(X)$ be such that $\bigcup_{S \in I} S \in \mathscr{P}_{fin}(X)$, and suppose $x \in \bigcap_{S \in I} C(S)$. Then for each $S \in I$, $y > x$ for no $y \in S$, so $x \in C(\bigcup_{S \in I} S)$.

($\Leftarrow$) Suppose C satisfies Property α and Property γ. We must show that $>^C$ M-rationalizes C. Again, in light of the proof of Theorem 3.8, we only show that $M(S, >^C) \subseteq C(S)$ for all $S \in \mathscr{P}_{fin}(X)$. So let $S \in \mathscr{P}_{fin}(X)$, and suppose $x \in M(S, >^C)$. Then $y >^C x$ for no $y \in S$ and therefore by Regularity $x \in \bigcap_{y \in S} C(\{x, y\})$, so by Property γ, $x \in C(S)$. □

Corollary 3.10 (cf. [19], p. 28)**.** *A regular choice function C is M-rational if and only if $>^C$ uniquely M-rationalizes C.*

Proof. The direction from right to left is trivial. For the other direction, observe that by Theorem 3.9, if C is M-rational, then C satisfies Property α and Property γ, so by the proof of Theorem 3.9, $>^C$ M-rationalizes C, whence by Proposition 3.4 $>^C$ uniquely M-rationalizes C. □

We will see below that Theorem 3.9 can be generalized to a larger class of choice functions. Indeed, we will see that this result holds for choice functions that *fail* to be regular.

Many important results involving the role of Chernoff's axiom presuppose that the choice functions used are regular. As Sen points out, Fishburn, Blair, and Suzumura seem to think that Property α guarantees that the base relation is acyclic. But it is easy to see that this is incorrect, for it is Regularity that corresponds to acyclicity of the base relation, and Property α is independent of Regularity.

4. Choice Functions for Take The Best

The first step in the definition of a choice function for Gigerenzer's heuristics is to articulate a notion of binary preference for a Take The Best frame. We offered two such notions of preference in Definition 2.5. Thus, we are now in a position to define choice functions for Gigerenzer's heuristics.

Definition 4.1. Let $\mathfrak{H} = (X, (Cue_i)_{i<n}, Ran, \succ)$ be an Original Take The Best model. We define a choice function $C_{\mathfrak{H}}$ on $(X, \mathscr{P}_{fin}(X))$ by setting for every $S \in \mathscr{P}_{fin}(X)$, $C_{\mathfrak{H}}(S) := M(S, \succ)$. We thereby call $C_{\mathfrak{H}}$ an *Original Take The Best choice function*, or an *OTB choice function* for short, and we say that $C_{\mathfrak{H}}$ is based on $\mathfrak{H}$.

Definition 4.2. Let $\mathfrak{H} = (X, (Cue_i)_{i<n}, Ran, >)$ be a Take The Best model. We define a choice function $C_{\mathfrak{H}}$ on $(X, \mathscr{P}_{fin}(X))$ by setting for every $S \in \mathscr{P}_{fin}(X)$, $C_{\mathfrak{H}}(S) := M(S, >)$. We call $C_{\mathfrak{H}}$ a *Take The Best choice function*, or just a *TB choice function* for short, and we say that $C_{\mathfrak{H}}$ is based on $\mathfrak{H}$.

To illustrate the use of the choice functions defined above, let us return for a moment to the population demographics task described earlier. It is clear that we can interpret the relation $>$ as 'has a larger population than.' This binary relation is *revealed* by two-alternative-choice tasks of the sort we presented earlier (Hamburg or Cologne?). Now, when we construct a choice function as in one of the definitions above, we go beyond these two-alternative-choice tasks in such a way that one can ask a question such as the following:

Which city has the largest population?
(a) Hamburg
(b) Cologne
(c) Munich

We can thereby represent choices from sets of cardinality greater than two. We believe that this is reasonable extension of Gigerenzer's program. The extension is constructed in terms of an underlying binary preference relation which is determined by a satisficing algorithm.

Of course, in order to construct a choice function for triples and larger sets, we maximize the underlying binary preference relation. One might argue that this does not satisfy the requirement (satisfied by the probabilistic mental models adopted by Gigerenzer) that inductive inference is carried out by a satisficing algorithm (in Simon's sense). Nevertheless, working with maximization seems to be the first obvious step toward extending the algorithm beyond two-alternative-choice tasks.

We will see that this extension requires the introduction of important changes into the traditional theory of choice. It seems that any 'fast and frugal' extension of Take The Best (or Original Take The Best) will require (at least) the introduction of identical changes. In this article we therefore limit ourselves to the study of the aforementioned extension by means of maximization. We present, nevertheless, the sketch of a 'fast and frugal' extension in the last section of this article (this extension is analyzed in detail in a companion article).

Let's focus now on the structure of the choice functions for take the best we just defined. Obviously, an OTB or TB choice function will not in general be a regular choice function. In fact, even a decisive Original Take The Best model may induce an OTB choice function that is not regular, as we saw in Example 2.7. We must therefore develop a theory of choice functions that does not impose Regularity in order to find a representation of the choice functions induced by Gigerenzer's algorithms.

We can add to the results of the previous section a generalization linking choice functions to Property α and Property γ. Indeed, Sen's result need not require Regularity. To illustrate why this is so, let us first define a somewhat peculiar preference relation.

Definition 4.3 (Preference). Let C be a choice function on a choice space $(X, \mathscr{P}_{fin}(X))$. We define a binary relation $\gtrdot^C$ on X by setting

$$\gtrdot^C := \{(x, y) \in X \times X : y \notin C(\{x, y\})\}$$

We then have the following general result.

Theorem 4.4. *A choice function C on $(X, \mathscr{P}_{fin}(X))$ is M-rational if and only if it satisfies Property α and Property γ.*

Proof.

($\Rightarrow$) This direction proceeds as in the proofs of Theorem 3.8 and Theorem 3.9.

($\Leftarrow$) Suppose C satisfies Property α and Property γ. We show that $\gtrdot^C$ M-rationalizes C. Let $S \in \mathscr{P}_{fin}(X)$. We first show that $M(S, \gtrdot^C) \subseteq C(S)$. Suppose $x \in M(S, \gtrdot^C)$. Then $y \gtrdot^C x$ for no $y \in S$ and therefore $x \in \bigcap_{y \in S} C(\{x, y\})$, so by Property γ, $x \in C(S)$. Now to show that $C(S) \subseteq M(S, \gtrdot^C)$, suppose $x \in C(S)$. Then by Property α, for every $y \in S$, $x \in C(\{x, y\})$ and so not $y \gtrdot^C x$, whereby $x \in M(S, \gtrdot^C)$. □

One may have observed that this result demands very little of a binary relation that rationalizes a choice function. To see why, consider a choice space $(X, \mathscr{P}_{fin}(X))$, where $X :=$

$\{x, y, z\}$, and a choice function C on $(X, \mathscr{P}_{fin}(X))$ defined by setting $C(\{x, y, z\}) := \emptyset$ and

$$C(\{x\}) := \{x\} \quad C(\{y\}) := \{y\} \quad C(\{z\}) := \emptyset$$
$$C(\{x, z\}) := \{x\} \quad C(\{y, z\}) := \{y\} \quad C(\{x, y\}) := \emptyset.$$

It is easy check that C thus defined satisfies Property α and Property γ. Observe that among other things, the relation $\gtrdot^C$ is neither asymmetric nor irreflexive.

The lesson to be drawn from this is that although a generalization of the results of the previous section would involve conditions weaker than Regularity, a better generalization would ensure that a binary relation that M-rationalizes a choice function is asymmetric or at least irreflexive. We can guarantee irreflexivity with the following property:

Property ρ. For every $x \in X$ such that $\{x\} \in \mathscr{S}$, $C(\{x\}) = \{x\}$

We can also guarantee asymmetry with the following property:

Property σ. For every $x, y \in X$ such that $\{x, y\} \in \mathscr{S}$, $C(\{x, y\}) \neq \emptyset$.

Observe that as one should expect, Property σ entails Property ρ. We obtain our first better result.

Theorem 4.5. *A choice function C on $(X, \mathscr{P}_{fin}(X))$ is irreflexive M-rational if and only if it satisfies Property α, Property γ, and Property ρ.*

Proof. The proof here proceeds as in the proof of Theorem 4.4. Clearly C is irreflexive M-rational only if it satisfies Property ρ as well as Property α and Property γ. If C satisfies Property ρ, $\gtrdot^C$ is irreflexive, and if C also satisfies Property α and Property γ, $\gtrdot^C$ M-rationalizes C. □

We then have a corollary similar to Corollary 3.10.

Corollary 4.6. *A choice function C on $(X, \mathscr{P}_{fin}(X))$ satisfying Property ρ is M-rational if and only if $\gtrdot^C$ uniquely M-rationalizes C.*

But of course, an M-rational choice function satisfying Property ρ is not necessarily rationalized by $>^C$, which guarantees asymmetry. Yet both OTB are TB choice functions are M-rationalized by asymmetric binary relations, so we should seek a representation result that ensures as much. The next result ensures that any binary relation that M-rationalizes C is asymmetric, and of course, for this $>^C$ meets the task.

Theorem 4.7. *A choice function C on $(X, \mathscr{P}_{fin}(X))$ is asymmetric M-rational if and only if it satisfies Property α, Property γ, and Property σ.*

Proof. As before, the proof proceeds as in the proof of Theorem 4.4, and obviously C is asymmetric M-rational only if it satisfies Property σ in addition to Property α and Property γ. For the converse, observe that if C satisfies Property σ, then $x \gtrdot^C y$ if and only if $x \in C(\{x, y\})$ and $y \notin C(\{x, y\})$, and the latter holds just in case $x >^C y$. Of course, if C also satisfies Property α and Property γ, $\gtrdot^C$ and so $>^C$ M-rationalizes C. □

We again have a corollary.

Corollary 4.8. *A choice function C on $(X, \mathscr{P}_{fin}(X))$ satisfying Property σ is M-rational if and only if $>^C$ uniquely M-rationalizes C.*

It also follows that a choice function C satisfying Property σ is M-rational just in case $\gtrdot^C$ M-rationalizes C. Moreover, OTB and TB choice functions are at most singleton-valued. That is, if $C_{\mathfrak{H}}$ is an OTB or TB choice function, then for every $S \in \mathscr{P}_{fin}(X)$, if $C_{\mathfrak{H}}(S) \neq \emptyset$, then $|C_{\mathfrak{H}}(S)| = 1$. (For an arbitrary set A, we write $|A|$ to denote the cardinality of A.) Why is this? This is so because every OTB or TB choice function is also connected M-rational, as indicated in Section 1 . We can guarantee connectivity with the following condition:

Property π. For every $S \in \mathscr{S}$, if $C(S) \neq \emptyset$, then $|C(S)| = 1$.

As one should expect, Property π is independent of Property σ, and in the presence of Property α, a choice function C on $(X, \mathscr{P}_{fin}(X))$ satisfies Property π if and only if for every $x, y \in X$, $|C(\{x, y\})| = 1$.

Now that we have briefly indicated how to generalize the results of the previous section, we present a result most relevant for our study of OTB and TB choice functions.

Theorem 4.9. *A choice function C is asymmetric, connected M-rational if and only if it satisfies Property α, Property γ, Property σ, and Property π.*

Proof. The direction from left to right is trivial. For the converse, observe if C satisfies Property α, Property γ, and Property σ, $>^C$ M-rationalizes C and is furthermore asymmetric. If C also satisfies Property π, clearly $>^C$ is connected. □

Of course, OTB and TB choice functions satisfy Property α, Property γ, Property σ, and Property π. We thereby have the following theorem.

Theorem 4.10. *Every OTB or TB choice function satisfies Property α, Property γ, Property σ, and Property π.*

Indeed, the converse holds for OTB and TB choice functions.

Theorem 4.11. *Every choice function satisfying Property α, Property γ, Property σ, and Property π is both an OTB and TB choice function.*

What about OTB choice functions based on an Original Take The Best model that has no guessing? Somewhat surprisingly, such OTB choice functions can be classified by the same properties of the previous theorem.

Theorem 4.12. *Every choice function C satisfying Property α, Property γ, Property σ, and Property π is an OTB choice function $C_{\mathfrak{H}}$ based on a decisive Original Take The Best model $\mathfrak{H}$.*

Can we produce a similar representation for a Take The Best model? It is easy to see that if $\mathfrak{H}$ is a discriminating Take The Best model, then $\mathfrak{H}$ is asymmetric, connected, and transitive, whereby it follows that $\mathfrak{H}$ is also negatively transitive and acyclic.

Lemma 4.13. *If $\mathfrak{H}$ is a discriminating Take The Best model, then $>$ is transitive.*

Proof. Let $x, y, z \in X$ be such that $x > y$ and $y > z$. Let i be the least index for which $x >_i y$, and let j be the least index for which $y >_j z$. First, assume $i < j$. Then it must be the case that $Cue_i(z) \in \{-, ?\}$, for otherwise, if $Cue_i(z) = +$, then $>_i$ is discriminating between z and y, yielding a contradiction; thus, $Cue_i(z) \in \{-, ?\}$. Furthermore, i is the least index for which Cue_i is discriminating between x and z, for otherwise, either j is not the least index for which Cue_j is discriminating between y and z or i is not the least index for which Cue_i is discriminating between x and y. Thus, $x > z$. A similar argument shows that if $j < i$, then $x > z$. □

Choice functions based on a discriminating Take The Best model are 'super' rational, and in addition to Property π, satisfy the following conditions:

Property β. For every $S, T \in \mathscr{S}$, if $S \subseteq T$ and $C(S) \cap C(T) \neq \emptyset$, then $C(S) \subseteq C(T)$.

Aizerman's Property. For every $S, T \in \mathscr{S}$, if $S \subseteq T$ and $C(T) \subseteq S$, then $C(T) \subseteq C(S)$.
Property ι. For every $S \in \mathscr{S}$, $|C(S)| = 1$.

We then have the following general result.

Theorem 4.14. *Let C be a choice function. Then the following are equivalent:*

(i) *C is asymmetric, transitive, connected M-rational.*
(ii) *C is asymmetric, negatively transitive, connected M-rational.*
(iii) *C is acyclic, connected M-rational.*
(iv) *C satisfies Property α and Property ι.*
(v) *C is regular and satisfies Property α and Property π.*
(vi) *C is regular and satisfies Property α, Property γ, and Property π.*
(vii) *C is regular and satisfies Property α, Property β, and Property π.*
(viii) *C is regular and satisfies Property α, Aizerman's Property, and Property π.*

Proof. Left to the reader. □

Such properties characterize a Take The Best model for which there is no guessing.

Theorem 4.15. *Every TB choice function $C_{\mathfrak{H}}$ based on a discriminating Take The Best model $\mathfrak{H}$ satisfies Property α and Property ι.*

Theorem 4.16. *Every choice function C satisfying Property α and Property ι is a TB choice function $C_{\mathfrak{H}}$ based on a discriminating Take The Best model $\mathfrak{H}$.*

Proof. Let C be a choice function satisfying Property α and Property ι. We first define a sequence of sets recursively. In the following, let $n := |X|$. Set $X_0 := C(X)$. Then assuming $X_1, \ldots, X_{m-1}$ are defined, set

$$X_m \quad := \quad \begin{cases} C(\bigcap_{i<m}(X \setminus X_i)) & \text{if } \bigcap_{i<m}(X \setminus X_i) \neq \emptyset \\ \emptyset & \text{otherwise.} \end{cases}$$

Clearly $\{X_i : i < n\}$ is a partition of X. We now define a collection of cues $(Cue_i)_{i<n}$. Define Cue_0 by setting $Cue_0(x) := +$ for every $x \in X$. Then for each i with $0 < i < n$, define Cue_i by setting for every $x \in X$,

$$Cue_i(x) \quad := \quad \begin{cases} + & \text{if } x \in \bigcup_{j<i} X_j \\ ? & \text{otherwise.} \end{cases}$$

Now let $\mathfrak{H} := (X, (Cue_i)_{i<n}, Ran, >)$, where $>$ is defined as in Definition 2.5 and Ran is picked arbitrarily. It is easy to verify that $\mathfrak{H}$ is a discriminating Take The Best model and that $> = >^C$, whence $C = C_{\mathfrak{H}}$, as desired. □

5. Conclusion and Discussion

There is an on-going discussion about the psychological plausibility of probabilistic mental models in general and Take The Best in particular (see [6] and [8]). Much of this discussion centers on the methods needed to obtain a cue validity ordering required to implement Take The Best. Some critics have suggested, for example, that cue validities cannot be computed because memory itself does not encode missing information. According to [8], this view is misinformed and the relevant cue orderings can arise from evolutionary, social, or individual learning.

It seem obvious that the plausibility of Take The Best very much depends on finding computationally feasible ways of determining cue validity orderings. In this article we presuppose that Gigerenzer and his colleagues are right and that the relevant cue validity orderings are computable. Then the main remaining problem is how to use cue validity orderings. Of course, it is important to take note that an Original Take The Best model and a Take The Best model are two fundamentally different ways of using the information provided by cues.

We saw that an Original Take The Best model, even without guessing, is compatible with violations of transitivity and the existence of preference cycles. On the other side of the spectrum, we have seen that the discriminating Take The Best model is particularly well behaved. The ordering that this model induces obeys transitivity, and the corresponding choice function obeys not only classical coherence constraints like Property α and Property γ but also other important constraints like Aizerman's Property and Property β.

Gigerenzer compares both heuristics in [7, p. 194-195]. He does not recommend one in particular but he mentions that it has been demonstrated that there are systematic intransitivities resulting from incommensurability along one dimension between two biological systems. This is the example he has in mind:

> Imagine three species *a*, *b* and *c*. Species *a* inhabits both water and land; species *b* inhabits both water and air. Therefore the two only compete in water, where species *a* defeats species *b*. Species *c* inhabits land and air, so it only competes with *b* in the air, where it is defeated by *b*. Finally, when *a* and *c* meet, it is only on land, and here, *c* is in its element and defeats *a*. A linear model that estimates some value for the combative strength of each species independently of the species with which it is competing would fail to capture this non-transitive cycle.

Gigerenzer seems to suggest that models that do not allow for the possibility of non-transitive cycles would not be rich enough to capture intransitivities exhibited by biological systems. This argument seems to favor the use of the Original Take The Best heuristic. But the argument based on the example is nevertheless notoriously weak. The non-transitive cycle appears only if one adopts a particularly poor representational strategy. Gigerenzer seems to argue that there are intrinsic cycles out there in the world to be captured by our models. Therefore, the imposition of transitivity on a model allegedly limits its representational capacity. But this is not the case. The existence of cycles is relative to a choice of representational language (where the predicate 'defeats' is not relativized to air, land or water). The cycle vanishes if the language is rich enough to represent the example more thoroughly.

On the other hand, one might argue that the agent exhibiting non-transitive preferences could be threatened by a smart bookie who can always take advantage of him. These pragmatic arguments, nevertheless, are plagued by all sort of controversial assumptions that limit their use.

Even if one is convinced that there is no reason to use the stricter stopping rule characteristic of the Original Take The Best heuristic[3], we remind the reader that the use of guessing can reintroduce non-transitive cycles. So it seems that a general theory of choice functions for Take The Best should allow for the possibility of cycles and therefore should abandon Regularity.

Of course, a theory of choice functions of this sort does not have a prescriptive or normative role, but a descriptive role. Its sole function is not to impose rationality constraints on choice but to faithfully register patterns of choice permitted by the algorithm. As we

[3]For example, if one thinks that the stopping rule used in Take The Best is psychologically more adequate.

argued above, the theory that thus arises still retains some of the central coherence constraints of normative theories. But the interpretation of these constraints is different in the context of the descriptive theory. The coherence constraints (like Property α) reflect regularities verified by the choice functions that correspond to the algorithm, which, in turn, is taken as an epistemological primitive.

One of the central points that Gigerenzer wants to make is that fast and frugal methods, when adapted, can be as accurate or more accurate than linear models that integrate information. But he seems to suggest as well that speed and frugality cannot be reconciled with the traditional norms of rationality, like transitivity. [4] As we showed in this article, Gigerenzer has failed to make a strong case for questioning this view. Indeed, there are frugal methods that are fairly well behaved from the normative point of view (like a discriminating Take The Best model).

5.1. **Future work.** In section 3 we noted that it would be nice to have a 'fast and frugal' extension of TB. We are working on such an extension in a companion article [3]. Here is the basic idea of the extension: when an agent has to decide what is admissible from a set S of cardinality larger than two the decision involves first an act of *framing* where the set is ordered producing a list. So, if the set has three elements the initial stage consists on producing a list:

$$L(S) = (a_1, a_2, a_3)$$

After the set S has been framed as a list, the list is used to pick an element as follows: one starts with the first pair (a_1, a_2) and compares the two elements (by using the binary relation $<$ that we have used in this article). Say that the dominant element is a_1. Then a_1 is compared with the third element a_3. We take the dominant element from this pair and the process terminates by selecting this dominant element (say a_3).

This type of choice has been studied in a recent article by A. Rubinstein and Y. Salant [15]. The authors call it *successive choice*. As a matter of fact [15] offers an interesting model of choice from lists that is obviously relevant for our extension of TB.

Among other things, Rubinstein and Salant discuss various axioms that apply to choice from lists. A salient one is a version of the independence of irrelevant alternatives (another name of the condition we called α above). They call this axiom 'List Independence of Irrelevant Alternatives' (LIIA):

LIIA A choice function from lists C satisfies LIIA for every list $(a_1, \ldots a_k)$, if $C(a_1, \ldots a_k)$ $= a_i$, then $C(a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_k)$, for all $1 \leq j \leq k$, $j \neq i$.

The axiom says that deleting an element that is not chosen from a list does not alter the choice. They then prove a representation result for choice from lists in terms of this axiom (in fact in terms of a condition that is equivalent to LIIA).

As Rubinstein and Salant remark in [15] not every successive choice function satisfies LIIA. In our case, when the underlying preference relation allows for cycles, it is easy to see that the choice function induced by our extension of Gigerenzer's algorithm fails to obey LIIA.

When the underlying preference relation does obey transitivity the extended choice function is better behaved and we can use Rubinstein and Salant's main representation result to characterize it. In any case, it is clear that that this extension does meet the requirement of representing inductive inference via a satisficing algorithm (it turns out that satisficing itself is a particular type of choice from lists).

[4] In fact, Gigerenzer's interest is to question the view—that he sees as common and widespread—that *only* 'rational' methods are accurate.

There are many possible areas of application of the theory sketched here. One of these applications is the area of belief change. One can study, for example, the notion of *contraction* of a sentence A from a set of sentences K, which consist on effectively eliminating A from K. Notice that in order to do so it is not enough to just delete A because the sentence might be implied by other sentences in K. So one has to *choose* what elements of K one wants to delete in order to contract A from K. Recently there has been some work studying the general properties that an operator of contraction should have. And there has been some work as well exploring systematic relations between standard axioms in the theory of choice functions (conditions like α, etc) and salient axioms in the theory of contraction. So, it seems that an obvious thing one can do is to consider the constraints on contraction functions imposed by bounded conditions on choice. The corresponding conditions can axiomatize a bounded notion of belief change where one choses among possible contractions in a fast and frugal way.

We consider the application sketched above in a second companion article. We also consider there in more detail some foundational issues related to the model presented in this article (issues related to the reasons for looking at the choice functions induced by fast and frugal methods).

References

[1] M. Aizerman. New problems in the general choice theory: Review of a research trend. *Social Choice and Welfare*, 2(4):235–282, December 1985.

[2] M. Aizerman and A. Malishevksi. General theory of best varients choice: Some aspects. *IEEE Transactions of Automatic Control*, 26(5):1030–1040, October 1981.

[3] H. Arló-Costa and A.P. Pedersen. A fast and frugal extension of Gigerenzer's Take the Best. Manuscript, Carnegie Mellon University, 2008.

[4] J.L. Bermúdez. Rationality, logic and fast and frugal heuristics. Forthcoming in *Behavioral and Brain Sciences*.

[5] M.S. Dawkins. *Unravelling Animal Behaviour*. Longman, second edition, 1995.

[6] A.M. Dougherty, M.R. Franco-Watkins and R. Thomas. Psychological plausibility of the theory of probabilistic mental models and the fast and frugal heuristics. *Psychological Review*, 115(1):199, 213, 2008.

[7] G. Gigerenzer. *Adaptive Thinking: Rationality in the Real World*. Oxford University Press, 2000.

[8] U. Gigerenzer, G. Hoffrage and D.G. Goldstein. Fast and frugal heuristics are plausible models of cognition: Reply to Dougherty, Franco-Watkins, and Thomas (2008). *Psychological Review*, 115(1):230–239, 2008.

[9] S.O. Hansson. *A Textbook of Belief Dynamics*. Kluwer Academic Publishers, 1999.

[10] D. Kahneman. A perspective on judgment and choice: Mapping bounded rationality. *American Psychologist*, 58(9):697–720, September 2003.

[11] D. Kahneman and A. Tversky. Judgment under uncertainty: Heuristics and biases. *Science*, 185:1124–1131, 1974.

[12] H. Moulin. Choice functions over a finite set: A summary. *Social Choice and Welfare*, 2(2):147–160, September 1985.

[13] H. Rott. Belief contraction in the context of the general theory of rational choice. *Journal of Symbolic Logic*, 58(4):1426–1450, December 1993.

[14] H. Rott. *Change, Choice and Inference: A Study of Belief Revision and Non-Monotonic Reasoning*. Oxford University Press, 2001.

[15] A. Rubinstein and Y. Salant. A model of choice from lists. *Theoretical Economics*, 45:3–17, 2006.

[16] A.K. Sen. Choice functions and revealed preference. *The Review of Economic Studies*, 38(3):307–317, July 1971.

[17] A.K. Sen. Social choice theory: A re-examination. *Econometrica*, 45(1):53–88, January 1977.

[18] H.A. Simon. Invariants of human behavior. *Annual Review of Psychology*, 41:1–20, January 1990.

[19] K. Suzumura. *Rational Choice, Collective Decisions, and Social Welfare*. Cambridge University Press, 1983.